Linear algebra review

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1 Eigenvalues and eigenvectors

Definition 1. A $d \times d$ matrix $\mathbf{M}$ has *eigenvalue* λ if there is a d-dimensional vector $\mathbf{u} \neq \mathbf{0}$ for which $\mathbf{M}\mathbf{u} = \lambda\mathbf{u}$. This $\mathbf{u}$ is the *eigenvector* corresponding to λ.

In other words, the linear transformation $\mathbf{M}$ maps vector $\mathbf{u}$ into the same direction. It is interesting that *any* linear transformation necessarily has directional fixed points of this kind. The following chain of implications helps in understanding this:

$$\begin{aligned}
& \lambda \text{ is an eigenvalue of } \mathbf{M} \\
\Leftrightarrow\ & \text{there exists } \mathbf{u} \neq \mathbf{0} \text{ with } \mathbf{M}\mathbf{u} = \lambda\mathbf{u} \\
\Leftrightarrow\ & \text{there exists } \mathbf{u} \neq \mathbf{0} \text{ with } (\mathbf{M} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0} \\
\Leftrightarrow\ & (\mathbf{M} - \lambda\mathbf{I}) \text{ is singular (that is, not invertible)} \\
\Leftrightarrow\ & \det(\mathbf{M} - \lambda\mathbf{I}) = 0.
\end{aligned}$$

Now, $\det(\mathbf{M} - \lambda\mathbf{I})$ is a polynomial of degree d in λ. As such it has d roots (although some of them might be complex). This explains the existence of eigenvalues.

A case of great interest is when $\mathbf{M}$ is real-valued and symmetric, because then the eigenvalues are real.

Theorem 2. *Let $\mathbf{M}$ be any real symmetric $d \times d$ matrix. Then:*

1. *$\mathbf{M}$ has d real eigenvalues $\lambda_1, \ldots, \lambda_d$ (not necessarily distinct).*
2. *There is a set of d corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$ that constitute an orthonormal basis of $\mathbb{R}^d$, that is, $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$ for all i, j.*

2 Spectral decomposition

The spectral decomposition recasts a matrix in terms of its eigenvalues and eigenvectors. This representation turns out to be enormously useful.

Theorem 3. *Let $\mathbf{M}$ be a real symmetric $d \times d$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_d$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$. Then:*

1. $\mathbf{M} = \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_d \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{\text{call this } \mathbf{Q}} \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{pmatrix}}_{\mathbf{\Lambda}} \underbrace{\begin{pmatrix} \longleftarrow & \mathbf{u}_1 & \longrightarrow \\ \longleftarrow & \mathbf{u}_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{u}_d & \longrightarrow \end{pmatrix}}_{\mathbf{Q}^T}.$

2. $\mathbf{M} = \sum_{i=1}^d \lambda_i \mathbf{u}_i \mathbf{u}_i^T$.

Proof. A general proof strategy is to observe that $\mathbf{M}$ represents a linear transformation $\mathbf{x} \mapsto \mathbf{Mx}$ on $\mathbb{R}^d$, and as such, is completely determined by its behavior on *any* set of d linearly independent vectors. For instance, $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ are linearly independent, so any $d \times d$ matrix $\mathbf{N}$ that satisfies $\mathbf{N}\mathbf{u}_i = \mathbf{M}\mathbf{u}_i$ (for all i) is necessarily identical to $\mathbf{M}$.

Let's start by verifying (1). For practice, we'll do this two different ways.

Method One: For any i, we have

$$\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T\mathbf{u}_i \;=\; \mathbf{Q}\mathbf{\Lambda}\mathbf{e}_i \;=\; \mathbf{Q}\lambda_i\mathbf{e}_i \;=\; \lambda_i\mathbf{Q}\mathbf{e}_i \;=\; \lambda_i\mathbf{u}_i \;=\; \mathbf{M}\mathbf{u}_i.$$

Thus $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T = \mathbf{M}$.

Method Two: Since the $\mathbf{u}_i$ are orthonormal, we have $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Thus $\mathbf{Q}$ is invertible, with $\mathbf{Q}^{-1} = \mathbf{Q}^T$; whereupon $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. For any i,

$$\mathbf{Q}^T\mathbf{M}\mathbf{Q}\mathbf{e}_i \;=\; \mathbf{Q}^T\mathbf{M}\mathbf{u}_i \;=\; \mathbf{Q}^T\lambda_i\mathbf{u}_i \;=\; \lambda_i\mathbf{Q}^T\mathbf{u}_i \;=\; \lambda_i\mathbf{e}_i \;=\; \mathbf{\Lambda}\mathbf{e}_i.$$

Thus $\mathbf{\Lambda} = \mathbf{Q}^T\mathbf{M}\mathbf{Q}$, which implies $\mathbf{M} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$.

Now for (2). Again we use the same proof strategy. For any j,

$$\left(\sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^T\right)\mathbf{u}_j \;=\; \lambda_j\mathbf{u}_j \;=\; \mathbf{M}\mathbf{u}_j.$$

Hence $\mathbf{M} = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^T$. ∎

3 Positive semidefinite matrices

We now introduce an important subclass of real symmetric matrices.

Definition 4. A real symmetric $d \times d$ matrix $\mathbf{M}$ is *positive semidefinite* (denoted $\mathbf{M} \succcurlyeq 0$) if $\mathbf{z}^T\mathbf{M}\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathbb{R}^d$. It is *positive definite* (denoted $\mathbf{M} \succ 0$) if $\mathbf{z}^T\mathbf{M}\mathbf{z} > 0$ for all nonzero $\mathbf{z} \in \mathbb{R}^d$.

Example 5. Consider any random vector $X \in \mathbb{R}^d$, and let $\mu = \mathbb{E}X$ and $\mathbf{S} = \mathbb{E}[(X-\mu)(X-\mu)^T]$ denote its mean and covariance, respectively. Then $\mathbf{S} \succcurlyeq 0$ because for any $\mathbf{z} \in \mathbb{R}^d$,

$$\mathbf{z}^T\mathbf{S}\mathbf{z} \;=\; \mathbf{z}^T\mathbb{E}[(X-\mu)(X-\mu)^T]\mathbf{z} \;=\; \mathbb{E}[(\mathbf{z}^T(X-\mu))((X-\mu)^T\mathbf{z})] \;=\; \mathbb{E}[(\mathbf{z}\cdot(X-\mu))^2] \;\geq\; 0.$$

Positive (semi)definiteness is easily characterized in terms of eigenvalues.

Theorem 6. *Let $\mathbf{M}$ be a real symmetric $d \times d$ matrix. Then:*

1. *$\mathbf{M}$ is positive semidefinite iff all its eigenvalues $\lambda_i \geq 0$.*

2. *$\mathbf{M}$ is positive definite iff all its eigenvalues $\lambda_i > 0$.*

Proof. Let's prove (1) (the second is similar). Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\mathbf{M}$, with corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$.

First, suppose $\mathbf{M} \succcurlyeq 0$. Then for all i, $\lambda_i = \mathbf{u}_i^T\mathbf{M}\mathbf{u}_i \geq 0$.

Conversely, suppose that all the $\lambda_i \geq 0$. Then for any $\mathbf{z} \in \mathbb{R}^d$, we have

$$\mathbf{z}^T\mathbf{M}\mathbf{z} \;=\; \mathbf{z}^T\left(\sum_{i=1}^d \lambda_i \mathbf{u}_i \mathbf{u}_i^T\right)\mathbf{z} \;=\; \sum_{i=1}^d \lambda_i(\mathbf{z}\cdot\mathbf{u})^2 \;\geq\; 0.$$

∎

4 The Rayleigh quotient

One of the reasons why eigenvalues are so useful is that they constitute the optimal solution of a very basic quadratic optimization problem.

Theorem 7. *Let $\mathbf{M}$ be a real symmetric $d \times d$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, and corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$. Then:*

$$\max_{\|\mathbf{z}\|_2=1} \mathbf{z}^T \mathbf{M} \mathbf{z} \;=\; \max_{\mathbf{z} \neq \mathbf{0}} \frac{\mathbf{z}^T \mathbf{M} \mathbf{z}}{\mathbf{z}^T \mathbf{z}} \;=\; \lambda_1$$

$$\min_{\|\mathbf{z}\|_2=1} \mathbf{z}^T \mathbf{M} \mathbf{z} \;=\; \min_{\mathbf{z} \neq \mathbf{0}} \frac{\mathbf{z}^T \mathbf{M} \mathbf{z}}{\mathbf{z}^T \mathbf{z}} \;=\; \lambda_d$$

and these are realized at $\mathbf{z} = \mathbf{u}_1$ and $\mathbf{z} = \mathbf{u}_d$, respectively.

Proof. Denote the spectral decomposition by $\mathbf{M} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T$. Then:

$$\begin{aligned}
\max_{\mathbf{z} \neq \mathbf{0}} \frac{\mathbf{z}^T \mathbf{M} \mathbf{z}}{\mathbf{z}^T \mathbf{z}} &= \max_{\mathbf{z} \neq \mathbf{0}} \frac{\mathbf{z}^T \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T \mathbf{z}}{\mathbf{z}^T \mathbf{Q} \mathbf{Q}^T \mathbf{z}} \qquad (\text{since } \mathbf{Q}\mathbf{Q}^T = \mathbf{I}) \\
&= \max_{\mathbf{y} \neq \mathbf{0}} \frac{\mathbf{y}^T \mathbf{\Lambda} \mathbf{y}}{\mathbf{y}^T \mathbf{y}} \qquad (\text{writing } \mathbf{y} = \mathbf{Q}^T \mathbf{z}) \\
&= \max_{\mathbf{y} \neq \mathbf{0}} \frac{\lambda_1 y_1^2 + \cdots + \lambda_d y_d^2}{y_1^2 + \cdots + y_d^2} \;\leq\; \lambda_1,
\end{aligned}$$

where equality is attained in the last step when $\mathbf{y} = \mathbf{e}_1$, that is, $\mathbf{z} = \mathbf{Q}\mathbf{e}_1 = \mathbf{u}_1$. The argument for the minimum is identical. □

Example 8. Suppose random vector $X \in \mathbb{R}^d$ has mean μ and covariance matrix $\mathbf{M}$. Then $\mathbf{z}^T \mathbf{M} \mathbf{z}$ represents the variance of X in direction $\mathbf{z}$:

$$\text{var}(\mathbf{z}^T X) \;=\; \mathbb{E}[(\mathbf{z}^T (X - \mu))^2] \;=\; \mathbb{E}[\mathbf{z}^T (X - \mu)(X - \mu)^T \mathbf{z}] \;=\; \mathbf{z}^T \mathbf{M} \mathbf{z}.$$

Theorem 7 tells us that the direction of maximum variance is $\mathbf{u}_1$, and that of minimum variance is $\mathbf{u}_d$.

Continuing with this example, suppose that we are interested in the k-dimensional subspace (of $\mathbb{R}^d$) that has the most variance. How can this be formalized?

To start with, we will think of a linear projection from $\mathbb{R}^d$ to $\mathbb{R}^k$ as a function $\mathbf{x} \mapsto \mathbf{P}^T \mathbf{x}$, where $\mathbf{P}^T$ is a $k \times d$ matrix with $\mathbf{P}^T \mathbf{P} = \mathbf{I}_k$. The last condition simply says that the rows of the projection matrix are orthonormal.

When a random vector $X \in \mathbb{R}^d$ is subjected to such a projection, the resulting k-dimensional vector has covariance matrix

$$\text{cov}(\mathbf{P}^T X) \;=\; \mathbb{E}[\mathbf{P}^T (X - \mu)(X - \mu)^T \mathbf{P}] \;=\; \mathbf{P}^T \mathbf{M} \mathbf{P}.$$

Often we want to summarize the variance by just a single number rather than an entire matrix; in such cases, we typically use the *trace* of this matrix, and we write $\text{var}(\mathbf{P}^T X) = \text{tr}(\mathbf{P}^T \mathbf{M} \mathbf{P})$. This is also equal to $\mathbb{E}\|\mathbf{P}^T X - \mathbf{P}^T \mu\|_2^2$. With this terminology established, we can now determine the projection $\mathbf{P}^T$ that maximizes this variance.

Theorem 9. *Let $\mathbf{M}$ be a real symmetric $d \times d$ matrix as in Theorem 7. Pick any $k \leq d$.*

$$\max_{\mathbf{P} \in \mathbb{R}^{d \times k}, \mathbf{P}^T \mathbf{P} = \mathbf{I}} \text{tr}(\mathbf{P}^T \mathbf{M} \mathbf{P}) \;=\; \lambda_1 + \cdots + \lambda_k$$

$$\min_{\mathbf{P} \in \mathbb{R}^{d \times k}, \mathbf{P}^T \mathbf{P} = \mathbf{I}} \text{tr}(\mathbf{P}^T \mathbf{M} \mathbf{P}) \;=\; \lambda_{d-k+1} + \cdots + \lambda_d.$$

These are realized when the columns of $\mathbf{P}$ span the k-dimensional subspace spanned by $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{u}_{d-k+1}, \ldots, \mathbf{u}_d\}$, respectively.

Proof. We will prove the result for the maximum; the other case is symmetric. Let $\mathbf{p}_1, \ldots, \mathbf{p}_k$ denote the columns of $\mathbf{P}$. Then

$$\operatorname{tr}\left(\mathbf{P}^T\mathbf{M}\mathbf{P}\right) \;=\; \sum_{i=1}^{k} \mathbf{p}_i^T \mathbf{M} \mathbf{p}_i \;=\; \sum_{i=1}^{k} \mathbf{p}_i^T \left(\sum_{j=1}^{d} \lambda_j \mathbf{u}_j \mathbf{u}_j^T \right) \mathbf{p}_i \;=\; \sum_{j=1}^{d} \lambda_j \sum_{i=1}^{k} (\mathbf{p}_i \cdot \mathbf{u}_j)^2.$$

We will show that this quantity is at most $\lambda_1 + \cdots + \lambda_k$. To this end, let z_j denote $\sum_{i=1}^{k} (\mathbf{p}_i \cdot \mathbf{u}_j)^2$; clearly it is nonnegative. We will show that $\sum_j z_j = k$ and that each $z_j \leq 1$; the desired bound is then immediate.

First,

$$\sum_{j=1}^{d} z_j \;=\; \sum_{i=1}^{k}\sum_{j=1}^{d} (\mathbf{p}_i \cdot \mathbf{u}_j)^2 \;=\; \sum_{i=1}^{k}\sum_{j=1}^{d} \mathbf{p}_i^T \mathbf{u}_j \mathbf{u}_j^T \mathbf{p}_i \;=\; \sum_{i=1}^{k} \mathbf{p}_i^T \mathbf{Q}\mathbf{Q}^T \mathbf{p}_i \;=\; \sum_{i=1}^{k} \|\mathbf{p}_i\|_2^2 \;=\; k.$$

To upper-bound an individual z_j, start by extending the k orthonormal vectors $\mathbf{p}_1, \ldots, \mathbf{p}_k$ to a full orthonormal basis $\mathbf{p}_1, \ldots, \mathbf{p}_d$ of $\mathbb{R}^d$. Then

$$z_j \;=\; \sum_{i=1}^{k} (\mathbf{p}_i \cdot \mathbf{u}_j)^2 \;\leq\; \sum_{i=1}^{d} (\mathbf{p}_i \cdot \mathbf{u}_j)^2 \;=\; \sum_{i=1}^{d} \mathbf{u}_j^T \mathbf{p}_i \mathbf{p}_i^T \mathbf{u}_j \;=\; \|\mathbf{u}_j^T\|_2^2 \;=\; 1.$$

It then follows that

$$\operatorname{tr}\left(\mathbf{P}^T\mathbf{M}\mathbf{P}\right) \;=\; \sum_{j=1}^{d} \lambda_j z_j \;\leq\; \lambda_1 + \cdots + \lambda_k,$$

and equality holds when $\mathbf{p}_1, \ldots, \mathbf{p}_k$ span the same space as $\mathbf{u}_1, \ldots, \mathbf{u}_k$. □

5 Singular value decomposition

For any real symmetric $d \times d$ matrix $\mathbf{M}$, we can find its eigenvalues $\lambda_1, \cdots, \lambda_d$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_d$, and write

$$\mathbf{M} \;=\; \sum_{i=1}^{d} \lambda_i \mathbf{u}_i \mathbf{u}_i^T.$$

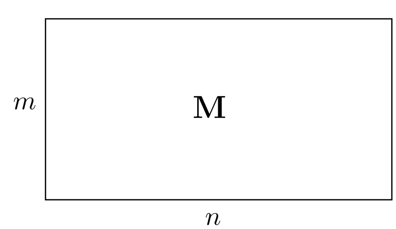
Assuming $|\lambda_1| \geq |\lambda_2| \geq \ldots |\lambda_d|$, the best rank-$k$ approximation to $\mathbf{M}$ is

$$\mathbf{M}_k \;=\; \sum_{i=1}^{k} \lambda_i \mathbf{u}_i \mathbf{u}_i^T,$$

in the sense that this minimizes $\|\mathbf{M} - \mathbf{M}_k\|_F^2$ over all rank-k matrices. (Here $\|\cdot\|_F$ denotes Frobenius norm; it is the same as l_2 norm if you imagine the matrix rearranged into a very long vector.)

In many applications, $\mathbf{M}_k$ is an adequate approximation of $\mathbf{M}$ even for fairly small values of k. And it is conveniently compact, of size $O(kd)$.

But what if we are dealing with a matrix $\mathbf{M}$ that is not square; say it is $m \times n$ with $m \leq n$:



To find a compact approximation in such cases, we look at $\mathbf{M}^T\mathbf{M}$ or $\mathbf{M}\mathbf{M}^T$, which are square. Eigendecompositions of these matrices lead to a good representation of $\mathbf{M}$.

5.1 The relationship between $\mathbf{M}\mathbf{M}^T$ and $\mathbf{M}^T\mathbf{M}$

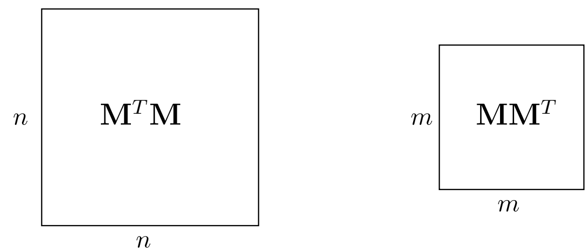
Lemma 10. *$\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$ are symmetric positive semidefinite matrices.*

Proof. We'll do $\mathbf{M}^T\mathbf{M}$; the other is similar. First off, it is symmetric:

$$(\mathbf{M}^T\mathbf{M})_{ij} \;=\; \sum_k (\mathbf{M}^T)_{ik}\mathbf{M}_{kj} \;=\; \sum_k \mathbf{M}_{ki}\mathbf{M}_{kj} \;=\; \sum_k (\mathbf{M}^T)_{jk}\mathbf{M}_{ki} \;=\; (\mathbf{M}^T\mathbf{M})_{ji}.$$

Next, $\mathbf{M}^T\mathbf{M} \succcurlyeq 0$ since for any $\mathbf{z} \in \mathbb{R}^n$, we have $\mathbf{z}^T\mathbf{M}^T\mathbf{M}\mathbf{z} = \|\mathbf{M}\mathbf{z}\|_2^2 \geq 0$. $\square$

Which one should we use, $\mathbf{M}^T\mathbf{M}$ or $\mathbf{M}\mathbf{M}^T$? Well, they are of different sizes, $n\times n$ and $m\times m$ respectively.



Ideally, we'd prefer to deal with the smaller of two, $\mathbf{M}\mathbf{M}^T$, especially since eigenvalue computations are expensive. Fortunately, it turns out the two matrices have the same (non-zero) eigenvalues!

Lemma 11. *If λ is an eigenvalue of $\mathbf{M}^T\mathbf{M}$ with eigenvector $\mathbf{u}$, then*

- **either:** (i) *λ is an eigenvalue of $\mathbf{M}\mathbf{M}^T$ with eigenvector $\mathbf{M}\mathbf{u}$,*
- **or** (ii) *$\lambda = 0$ and $\mathbf{M}\mathbf{u} = \mathbf{0}$.*

Proof. Say $\lambda \neq 0$; we'll prove that condition (i) holds. First of all, $\mathbf{M}^T\mathbf{M}\mathbf{u} = \lambda\mathbf{u} \neq \mathbf{0}$, so certainly $\mathbf{M}\mathbf{u} \neq \mathbf{0}$. It is an eigenvector of $\mathbf{M}\mathbf{M}^T$ with eigenvalue λ, since

$$\mathbf{M}\mathbf{M}^T(\mathbf{M}\mathbf{u}) \;=\; \mathbf{M}(\mathbf{M}^T\mathbf{M}\mathbf{u}) \;=\; \mathbf{M}(\lambda\mathbf{u}) \;=\; \lambda(\mathbf{M}\mathbf{u}).$$

Next, suppose $\lambda = 0$; we'll establish condition (ii). Notice that

$$\|\mathbf{M}\mathbf{u}\|_2^2 \;=\; \mathbf{u}^T\mathbf{M}^T\mathbf{M}\mathbf{u} \;=\; \mathbf{u}^T(\mathbf{M}^T\mathbf{M}\mathbf{u}) \;=\; \mathbf{u}^T(\lambda\mathbf{u}) \;=\; 0.$$

Thus it must be the case that $\mathbf{M}\mathbf{u} = \mathbf{0}$. $\square$

5.2 A spectral decomposition for rectangular matrices

Let's summarize the consequences of Lemma 11. We have two square matrices, a large one ($\mathbf{M}^T\mathbf{M}$) of size $n\times n$ and a smaller one ($\mathbf{M}\mathbf{M}^T$) of size $m\times m$. Let the eigenvalues of the large matrix be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, with corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. From the lemma, we know that at most m of the eigenvalues are nonzero.

The smaller matrix $\mathbf{M}\mathbf{M}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_m$, and corresponding orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. The lemma suggests that $\mathbf{v}_i = \mathbf{M}\mathbf{u}_i$; this is certainly a valid set of eigenvectors, but they are not necessarily normalized to unit length. So instead we set

$$\mathbf{v}_i \;=\; \frac{\mathbf{M}\mathbf{u}_i}{\|\mathbf{M}\mathbf{u}_i\|_2} \;=\; \frac{\mathbf{M}\mathbf{u}_i}{\sqrt{\mathbf{u}_i^T\mathbf{M}^T\mathbf{M}\mathbf{u}_i}} \;=\; \frac{\mathbf{M}\mathbf{u}_i}{\sqrt{\lambda_i}}.$$

This finally gives us the *singular value decomposition*, a spectral decomposition for general matrices.

Theorem 12. *Let* $\mathbf{M}$ *be a rectangular* $m \times n$ *matrix with* $m \leq n$. *Define* $\lambda_i, \mathbf{u}_i, \mathbf{v}_i$ *as above. Then*

$$\mathbf{M} \;=\; \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{\mathbf{Q}_1,\text{ size } m \times m} \underbrace{\left(\begin{array}{cccc|c} \sqrt{\lambda_1} & & & 0 & \\ & \sqrt{\lambda_2} & & & 0 \\ & & \ddots & & \\ 0 & & & \sqrt{\lambda_m} & \end{array}\right)}_{\mathbf{\Sigma},\text{ size } m \times n} \underbrace{\begin{pmatrix} \longleftarrow & \mathbf{u}_1 & \longrightarrow \\ \longleftarrow & \mathbf{u}_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{u}_n & \longrightarrow \end{pmatrix}}_{\mathbf{Q}_2^T,\text{ size } n \times n}.$$

Proof. We will check that $\mathbf{\Sigma} = \mathbf{Q}_1^T \mathbf{M} \mathbf{Q}_2$. By our proof strategy from Theorem 3, it is enough to verify that both sides have the same effect on $\mathbf{e}_i$ for all $1 \leq i \leq n$. For any such i,

$$\mathbf{Q}_1^T \mathbf{M} \mathbf{Q}_2 \mathbf{e}_i \;=\; \mathbf{Q}_1^T \mathbf{M} \mathbf{u}_i \;=\; \left\{ \begin{array}{ll} \mathbf{Q}_1^T \sqrt{\lambda_i} \mathbf{v}_i & \text{if } i \leq m \\ \mathbf{0} & \text{if } i > m \end{array} \right\} \;=\; \left\{ \begin{array}{ll} \sqrt{\lambda_i} \mathbf{e}_i & \text{if } i \leq m \\ \mathbf{0} & \text{if } i > m \end{array} \right\} \;=\; \mathbf{\Sigma} \mathbf{e}_i.$$

□

The alternative form of the singular value decomposition is

$$\mathbf{M} \;=\; \sum_{i=1}^{m} \sqrt{\lambda_i} \mathbf{v}_i \mathbf{u}_i^T,$$

which immediately yields a rank-k approximation

$$\mathbf{M}_k \;=\; \sum_{i=1}^{k} \sqrt{\lambda_i} \mathbf{v}_i \mathbf{u}_i^T.$$

As in the square case, $\mathbf{M}_k$ is the rank-k matrix that minimizes $\|\mathbf{M} - \mathbf{M}_k\|_F^2$.