COMS 4771 Lecture 5

1. Linear classifiers
2. Linearly separable instances.

Linear classifiers

Decision tree learning

Basic step in greedy decision tree learning (with axis-aligned splits in $\mathcal{X} = \mathbb{R}^d$):

$$\arg\min_h \frac{|S_{h,0}|}{|S|} u(S_{h,0}) + \frac{|S_{h,1}|}{|S|} u(S_{h,1})$$

where u is some uncertainty measure,

$$S_{h,0} = \{(x, y) \in S : h(x) = 0\}, \qquad S_{h,1} = \{(x, y) \in S : h(x) = 1\},$$

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Binary classification

When $\mathcal{Y} = \{-1, +1\}$, the prediction made in each split is either

$$\operatorname{sign}(x_i - t) \text{ or } -\operatorname{sign}(x_i - t) = \operatorname{sign}(-x_i + t)$$

Linear classifiers (for binary classification)

A natural generalization of axis-aligned threshold functions

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are **linear threshold functions** (or **linear classifiers**):

$$f_{\boldsymbol{w},t}(\boldsymbol{x}) = \mathrm{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle - t), \quad \boldsymbol{w} \in \mathbb{R}^d,\ t \in \mathbb{R}.$$

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Interpretation: does a weighted linear combination of input features exceed a threshold?

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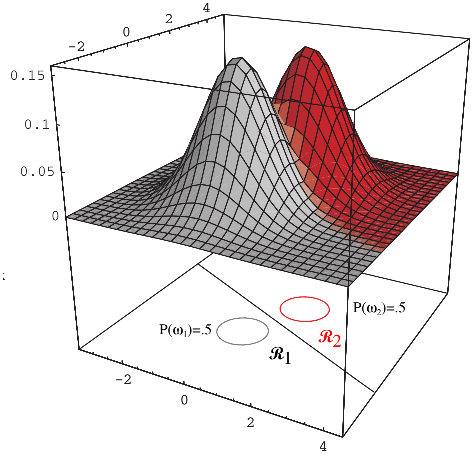
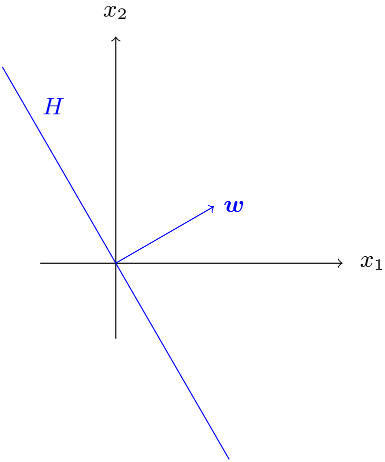
For now, only considering **binary classification**, where $\mathcal{Y} = \{-1, +1\}$.

We've seen these before: the (binary) Bayes classifier when class conditional densities are multivariate Gaussians with the same covariance.



HYPERPLANES

Geometric interpretation of linear classifiers



A **hyperplane** in $\mathbb{R}^d$ is a linear subspace of dimension $(d-1)$.

- A $\mathbb{R}^2$-hyperplane is a line.
- A $\mathbb{R}^3$-hyperplane is a plane.
- As a linear subspace, a hyperplane always contains the origin.

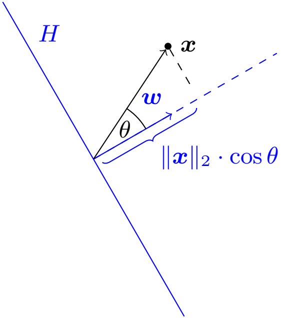
A hyperplane H can be specified by a (non-zero) **normal vector**.

The hyperplane with normal vector $\boldsymbol{w} \in \mathbb{R}^d$ is the set

$$H = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{w}, \boldsymbol{x} \rangle = 0 \right\}.$$

It becomes **oriented** if we pick a *particular* normal vector $\boldsymbol{w} \in \mathbb{R}^d$.

Which side of the hyperplane are we on?



Distance from the hyperplane

- The projection of $\boldsymbol{x}$ onto the direction of $\boldsymbol{w}$ has length $\dfrac{|\langle \boldsymbol{w}, \boldsymbol{x} \rangle|}{\|\boldsymbol{w}\|_2}$.

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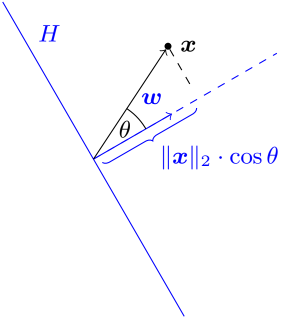

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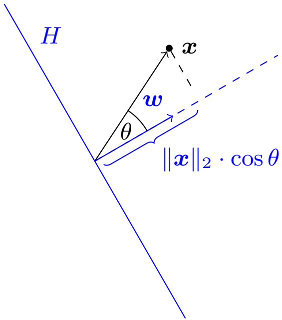
- The projection of $\boldsymbol{x}$ onto the direction of $\boldsymbol{w}$ has length $\dfrac{|\langle \boldsymbol{w}, \boldsymbol{x} \rangle|}{\|\boldsymbol{w}\|_2}$.
- Cosine rule: $\cos\theta = \dfrac{\langle \boldsymbol{w}, \boldsymbol{x} \rangle}{\|\boldsymbol{w}\|_2 \|\boldsymbol{x}\|_2}$.



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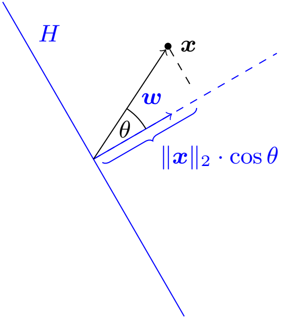

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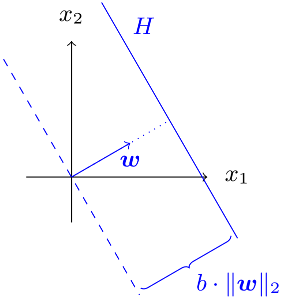
Which side of the hyperplane?

- The cosine satisfies $\cos\theta > 0$ iff $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- We can determine which side of the hyperplane H that $\boldsymbol{x}$ is on, using

$$\operatorname{sign}(\cos\theta) = \operatorname{sign}(\langle \boldsymbol{w}, \boldsymbol{x}\rangle).$$

Affine hyperplanes



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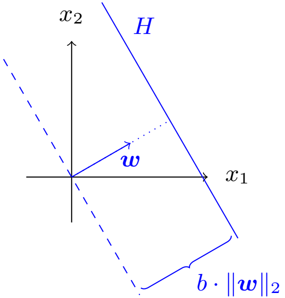

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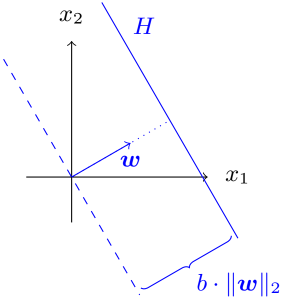

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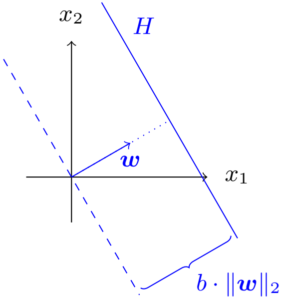

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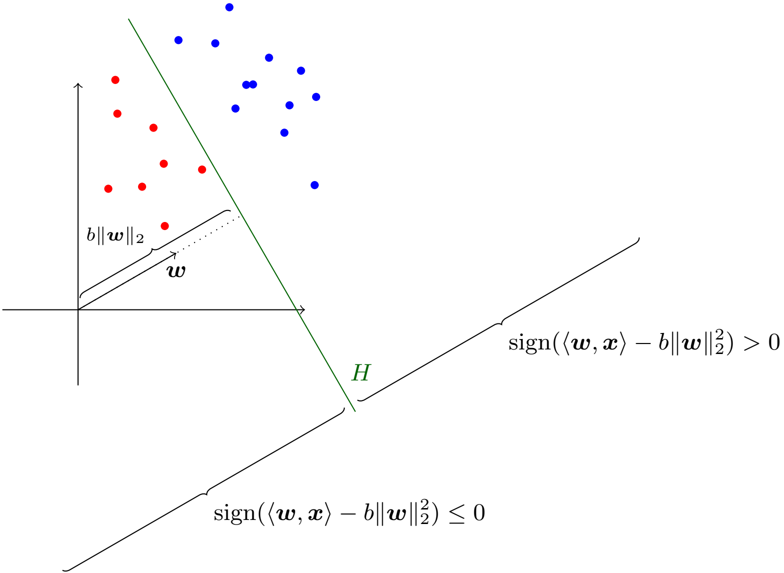
Which side of the affine hyperplane?

- We can determine which side of the affine hyperplane H that $\boldsymbol{x}$ is on using

$$\operatorname{sign}(\langle \boldsymbol{x}, \boldsymbol{w} \rangle - b\|\boldsymbol{w}\|_2^2).$$

side of affine hyperplane that $\boldsymbol{x}$ is on $\equiv$ linear classification of $\boldsymbol{x}$


$b\|\boldsymbol{w}\|_2$
$\boldsymbol{w}$
H
$\mathrm{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle - b\|\boldsymbol{w}\|_2^2) > 0$
$\mathrm{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle - b\|\boldsymbol{w}\|_2^2) \leq 0$

Learning linear classifiers

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Goal: learning algorithm for linear classifiers with low **error**:

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A natural approach is **"empirical risk minimization" (ERM)**: find a linear classifier $f_{\boldsymbol{w},t}$ with low training error (or **empirical risk**):

$$\begin{aligned}\operatorname*{arg\,min}_{\boldsymbol{w},t} \operatorname{err}(f_{\boldsymbol{w},t}, S) &= \operatorname*{arg\,min}_{\boldsymbol{w},t} \frac{1}{|S|} \sum_{(\boldsymbol{x},y)\in S} \mathbb{1}\{\operatorname{sign}(\langle \boldsymbol{w}, \boldsymbol{x}\rangle - t) \neq y\} \\ &= \operatorname*{arg\,min}_{\boldsymbol{w},t} \frac{1}{|S|} \sum_{(\boldsymbol{x},y)\in S} \mathbb{1}\{y(\langle \boldsymbol{w}, \boldsymbol{x}\rangle - t) \leq 0\}.\end{aligned}$$

Empirical risk minimization

Unforunately, this is not possible in general.

- The following problem is NP-hard:

  Given a set of labeled examples S in $\mathbb{R}^d \times \{\pm 1\}$ with the promise that there is a linear classifier with training error 0.01, find a linear classifier with training error ≤ 0.49.

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Plan:

1. Study the **linearly separable** instances: where there is a linear classifier with **zero training error**.
2. Study **convex loss functions**, which can be efficiently minimized, and how they are related to classification error.

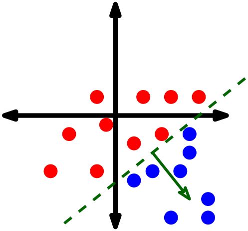
Linearly separable instances

Easy case: linearly separable data

Suppose there is a linear classifier with **zero training error** on S: for some $\boldsymbol{w}_\star \in \mathbb{R}^d$ and $t_\star \in \mathbb{R}$,

$$y(\langle \boldsymbol{w}_\star, \boldsymbol{x} \rangle - t_\star) > 0, \quad \text{for all } (x, y) \in S.$$

In this case, we say the training data is **linearly separable**.



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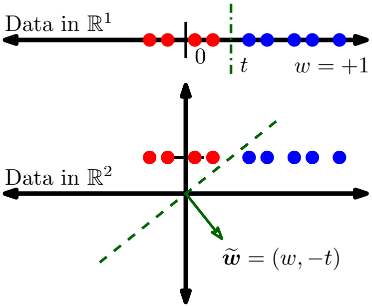
Proof: Let $\phi(\boldsymbol{x}) := (\boldsymbol{x}, 1)$—i.e., add a $(d+1)$-th coordinate that always takes value 1. For any $\boldsymbol{w} \in \mathbb{R}^d$ and $t \in \mathbb{R}$, let $\widetilde{\boldsymbol{w}} := (\boldsymbol{w}, -t)$.



Finding a homogeneous linear separator

Problem: given training data S in $\mathbb{R}^d \times \{\pm 1\}$, determine whether or not there exists $\boldsymbol{w} \in \mathbb{R}^d$

$$y\langle \boldsymbol{w}, \boldsymbol{x} \rangle > 0, \quad \text{for all } (\boldsymbol{x}, y) \in S;$$

(and find such a vector if one exists).

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If one exists, and the inequalities in fact hold with some non-negligible "margin" $\gamma > 0$:

$$y\langle \boldsymbol{w}, \boldsymbol{x} \rangle \geq \gamma, \quad \text{for all } (\boldsymbol{x}, y) \in S;$$

then there is a very **simple** algorithm that finds a solution: **Perceptron**.