COMS 4771 Lecture 3

1. Nearest neighbor classification.

Nearest neighbor classification

The optimal classifier

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The classifier $f\colon \mathcal{X} \to \mathcal{Y}$ with the smallest prediction error

$$\mathrm{err}(f) = \Pr[f(X) \neq Y]$$

is the **Bayes classifier**

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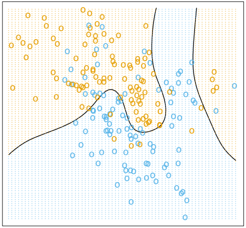
- **Last lecture**: use "generative" models to approximate $\Pr[Y = y | X = x]$.
- **This lecture**: directly approximate the **decision boundaries** of $f^\star$.



Nearest neighbor (NN) classifier

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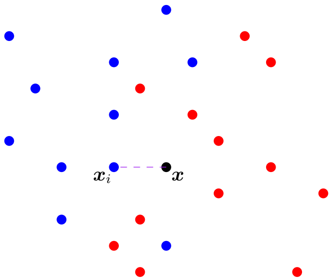
On input $\boldsymbol{x}$,

1. Let $\boldsymbol{x}_i$ be the point among $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ that is closest to $\boldsymbol{x}$.
2. Return y_i.



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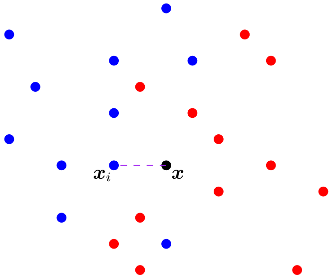

Question: how should we measure distance between points in $\mathcal{X}$?

DISTANCES

A default choice of distance for data in $\mathbb{R}^d$:

$$\text{Euclidean } (\ell_2) \text{ distance :} \quad \|\boldsymbol{u} - \boldsymbol{v}\|_2 := \sqrt{\sum_{i=1}^{d} (u_i - v_i)^2}.$$

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But there are many other options (which could be much better than ℓ_2) ...

- ℓ_p for $p \in [1, \infty]$:

$$\|\boldsymbol{u} - \boldsymbol{v}\|_p := \left(\sum_{i=1}^{d} |u_i - v_i|^p \right)^{1/p}.$$

- Edit distance (for strings): how add/delete/substitutions are requred to transform one string to the other.
- Shape distance (for images): figures out what "shape" is depicted in each image, then computes a distance based on how much "warping" is required to change one to the other.

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- **Observation**: First mistake (correct label is '2') might've been avoided by looking at three nearest neighbors (whose labels are '8', '2', '2') . . .

test point three nearest neighbors

k-NEAREST NEIGHBORS CLASSIFIER

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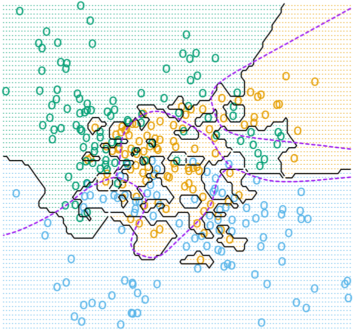
Example: OCR with k-NN classifier

k	1	3	5	7	9
$\mathrm{err}(\hat{f}_k, T)$	0.0309	0.0295	0.0312	0.0306	0.0341

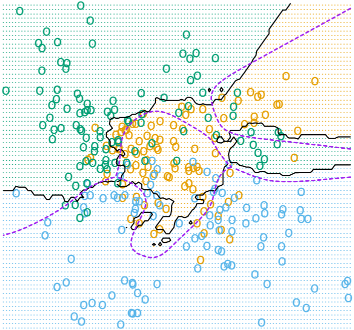
Effect of k

In general:

- Smaller $k \Rightarrow$ smaller training error. ($k = 1 \Rightarrow$ has zero training error.)
- Larger $k \Rightarrow$ predictions are more “stable” due to voting.



1-NN



15-NN

Purple dotted lines: Bayes classifier’s decision boundaries.
Black solid lines: k-NN’s decision boundaries.

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  More on this later in the course.

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k-**NN is consistent** provided that $k := k_n$ is chosen as an increasing but sublinear function of n:

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1-**NN is not consistent unless** $\mathrm{err}(f^\star) = 0$, although

$$\lim_{n\to\infty} \mathbb{E}\Big[\mathrm{err}(\hat{f}_1)\Big] \;\le\; 2\,\mathrm{err}(f^\star) \cdot \left(1 - \frac{K}{2(K-1)}\,\mathrm{err}(f^\star)\right).$$

Nearest neighbor search

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- **Alternatives**:
  1. Settle for an approximate nearest neighbor using *locality sensitive hash functions*.
  2. Store the n training data in a geometric data structure that permits fast NN queries.

(Informally:) A family $\mathcal{H}$ of hash functions from $\mathbb{R}^d$ to $\mathbb{Z}$ is a **locality-sensitive hash family** if

- For any points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{X}$ with $\|\boldsymbol{a} - \boldsymbol{b}\|_2 \ll \|\boldsymbol{a} - \boldsymbol{c}\|_2$,

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It turns out there are such hash families!

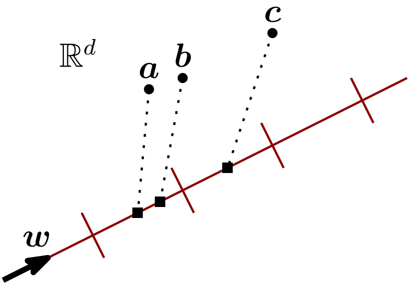
A locality sensitive hash family

A LSH family based on projections to one-dimensional subspaces

For $\boldsymbol{w} \in \mathbb{R}^d$ with $\|\boldsymbol{w}\|_2 = 1$, $r \in \{2^i : i \in \mathbb{Z}\}$, $s \in [0, r]$:

$$h_{\boldsymbol{w},r,s}(\boldsymbol{x}) := \left\lfloor \frac{\boldsymbol{w}^\top \boldsymbol{x} + s}{r} \right\rfloor .$$

- $\boldsymbol{w}$ determines the one-dimensional subspace,
- r determines a distance resolution, and
- s determines a shift of the bucket boundaries.



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Procedure:

- Select a hash function $h \in \mathcal{H}$ at random.

  (In practice, some parameters of the hash function, like r and s, may be tuned via hold-out or cross-validation.)
- Create pointer from buckets $j \in \mathbb{N}$ to points $\boldsymbol{x} \in S$ such that $h(\boldsymbol{x}) = j$.
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  (The bucket will generally contain far fewer than n points.)

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Do this with several hash functions from $\mathcal{H}$ to boost the chances that you find close neighbors.